International Journal of Theoretical Physics, Vol. 32, No. 12, 1993

Fermion Flows and Supersymmetry

R. L. Hudson[1]

Received November 30, 1992

Even fermionic stochastic flows are shown to be closely related to the mathematics of supersymmetry.

1. INTRODUCTION

The interface between differential geometry and stochastic calculus has been a fruitful source of mathematical inspiration. Recently attempts have been made to study this interface in a quantum or noncommutative context, both of the geometry and the stochastic calculus, based on the notion of a *quantum stochastic flow* (Hudson, 1990), which describes a stochastic flow on a noncommutative manifold in which the noise terms involve a noncommutative quantum stochastic calculus. Hitherto most attention has been devoted to the case where the latter is a bosonic theory; indeed it might be thought, in view of the unification of boson and fermion theories in Fock space (Hudson and Parthasarathy, 1986) that there was no distinctive fermionic alternative. In this paper we show that, despite the unification, the fermionic theory has a character all of its own, and is closely related to some recent theories of supersymmetry (Jaffe *et al.*, 1989).

The plan of the paper is as follows. In Section 2 we review quantum stochastic calculus in Fock space, including both boson and fermionic versions. In Section 3 we consider unitary stochastic evolutions driven by fermionic noise; this section is also largely of a review character. In Section 4 we consider fermionic flows and their structure maps. In contrast to a previous work (Hudson and Shepperson, 1992), we consider only flows comprising even Z_2-graded algebra homomorphisms; this simplification

[1]Mathematics Department, University of Nottingham, University Park, Nottingham NG7 2RD, England.

0020-7748/93/1200-2413$07.00/0

allows for a much more complete and satisfactory theory. Unitary perturbations are considered in Section 5. Finally, in Section 6 we compare the theory of fermion flows with the formalism of supersymmetry developed in Jaffe *et al.* (1989) (and in references therein).

In what follows, if h_1 and h_2 are Hilbert spaces, $h_1 \otimes h_2$ denotes the Hilbert space completion of the algebraic tensor product, which is denoted by $\underline{\otimes}$. Operators S and T defined on a common domain $\mathscr{E}$ are *mutually adjoint* if $\langle \psi_1, S\psi_2 \rangle = \langle T\psi_1, \psi_2 \rangle$ for arbitrary ψ_1 and $\psi_2 \in \mathscr{E}$. The Hilbert space inner product $\langle \cdot, \cdot \rangle$ is linear in the second variable.

2. QUANTUM STOCHASTIC CALCULUS
(Hudson and Parthasarathy, 1984; Parthasarathy, 1992)

The *boson Fock space* $\Gamma(h)$ over a Hilbert space h is conveniently characterized (up to isomorphism exchanging exponential vectors) as the Hilbert space generated by the *exponential vectors* $\psi(f), f \in h$, which satisfy

$$\langle \psi(f), \psi(g) \rangle = \exp\langle f, g \rangle$$

In what follows $h = L^2(\mathbb{R}_+)$. Corresponding to the natural direct sum decomposition

$$h = L^2(\mathbb{R}_+) = L^2[0, t] \oplus L^2(t, \infty) = h_t \oplus h^t$$

we make the identification

$$\Gamma(h) = \Gamma(h_t) \otimes \Gamma(h^t)$$

in which exponential vectors are product vectors; in particular, the dense *exponential domain* $\mathscr{E}$ spanned by the exponential vectors factorizes as an algebraic tensor product:

$$\mathscr{E} = \mathscr{E}_t \underline{\otimes} \mathscr{E}^t \tag{2.1}$$

The *creation*, *preservation*, and *annihilation processes* are the families of operators defined on the domain $\mathscr{E}$ by

$$A^\dagger(t)\psi(f) = \frac{d}{d\varepsilon}\psi(f + \varepsilon\chi_{[0,t]})\Big|_{\varepsilon=0}$$

$$\Lambda(t)\psi(f) = \frac{d}{d\varepsilon}\psi(e^{\varepsilon\chi_{[0,t]}}f)\Big|_{\varepsilon=0}$$

$$A(t)\psi(f) = \int_0^t f(s)\, ds\, \psi(f)$$

$\Lambda(t)$ is symmetric and $A(t)$, $A^\dagger(t)$ mutually adjoint on $\mathscr{E}$.

Quantum stochastic calculus renders meaningful operator-valued stochastic integral processes of the form

$$M(t) = \int_0^t (E\, d\Lambda + F\, dA^\dagger + G\, dA + H\, ds)$$

Here the integrands E, F, G, H are processes of operators defined on $\mathscr{E}$, together with their adjoints, which are *adapted* in the sense that, corresponding to (2.1),

$$E(t) = E_t \underline{\otimes} 1^t, \qquad F(t) = F_t \underline{\otimes} 1^t, \quad \text{etc.}$$

and which satisfy certain local square-integrability requirements. Then M is itself an adapted process whose adjoint process on $\mathscr{E}$ is given by

$$M^\dagger(t) = \int_0^t (E^\dagger\, d\Lambda + G^\dagger\, dA^\dagger + F^\dagger\, dA + H^\dagger\, ds) \tag{2.2}$$

The principal result of the theory is the *quantum Itô formula* for integration by parts:

$$\begin{aligned} &M_1(t)M_2(t) \\ &= \int_0^t \{(E_1 M_2 + M_1 E_2 + E_1 E_2)\, d\Lambda + (F_1 M_2 + M_1 F_2 + E_1 F_2)\, dA^\dagger \\ &\quad + (G_1 M_2 + M_1 G_2 + G_1 M_2)\, dA + (H_1 M_2 + M_1 H_2 + G_1 F_2)\, ds\} \end{aligned}$$

Here all products of unbounded operators are understood in the weak sense:

$$\langle \psi(f) \rangle, M_1(t) M_2(t) \psi(g) \rangle =: \langle M_1^\dagger(t) \psi(f), M_2(t) \psi(g) \rangle, \quad \text{etc.}$$

The quantum Itô formula is conveniently expressed in differential form as

$$d(M_1 M_2) = dM_1 \cdot M_2 + M_1 \cdot dM_2 + dM_1 \cdot dM_2 \tag{2.3}$$

with the convention that adapted processes commute with the basic differentials:

$$E\, d\Lambda = d\Lambda\, E \tag{2.4a}$$

$$E\, dA = dA\, E, \qquad E\, dA^\dagger = dA^\dagger\, E \tag{2.4b}$$

$$E\, ds = ds\, E \tag{2.4c}$$

and the *correction term* $dM_1 \cdot dM_2$ is evaluated from the multiplication table for the basic differentials

$$\begin{array}{c|cccc} & d\Lambda & dA^\dagger & dA & ds \\ \hline d\Lambda & d\Lambda & dA^\dagger & 0 & 0 \\ dA^\dagger & 0 & 0 & 0 & 0 \\ dA & dA & ds & 0 & 0 \\ ds & 0 & 0 & 0 & 0 \end{array} \tag{2.5}$$

The theory makes contact with classical Itô calculus through the Wiener–Segal isomorphism

$$\Gamma(h) \cong L^2 \quad \text{(Wiener measure)}$$

under which $A^\dagger + A$ becomes multiplication by Brownian motion, and multiplications by classically adapted processes are adapted processes in the quantum sense, stochastic integrals of which against $dA^\dagger + dA$ are multiplications by the Itô integral of the original classical process.

The usual boson creation and annihilation operators can be expressed as stochastic integrals (Cockroft and Hudson, 1977):

$$a(f) = \int \bar{f}\, dA, \qquad a^\dagger(f) = \int f\, dA^\dagger$$

in the sense that, for each $t \in \mathbb{R}_+$ and $f \in L^2(\mathbb{R}_+)$,

$$a(f\chi_{[0,t]}) = \int_0^t \bar{f}\, dA, \qquad a^\dagger(f\chi_{[0,t]}) = \int_0^t f\, dA^\dagger.$$

Introducing the *fermion creation and annihilation process*

$$B(t) = \int_0^t (-1)^\Lambda\, dA, \qquad b^\dagger(t) = \int_0^t (-1)^\Lambda\, dB$$

so that

$$dB = \Gamma\, dA, \qquad dB^\dagger = \Gamma\, dA^\dagger$$

where $\Gamma = (-1)^\Lambda$ is the *parity process*, and, since $\Gamma^2 = 1$,

$$dA = \Gamma\, dB, \qquad dA^\dagger = \Gamma\, dB^\dagger$$

we have a corresponding construction for fermion creation and annihilation operators (Hudson and Parthasarathy, 1986):

$$b(f) = \int \bar{f}\, dB = \int \bar{f}\Gamma\, dA, \qquad b^\dagger(f) = \int f\, dB^\dagger = \int f\Gamma\, dA^\dagger$$

Indeed these operators are bounded and satisfy the canonical anticommutation relations (CAR)

$$[b(f), b(g)]_+ = [b^\dagger(f), b^\dagger(g)]_+ = 0, \qquad [b(f), b^\dagger(g)]_+ = \langle f, g\rangle 1$$

and constitute the Fock representation of the CAR insofar as the vacuum $\psi(0)$ is annihilated by the $b(f)$ and is cyclic for the $b^\dagger(f)$. Fermionic theories of stochastic integration (Barnett *et al.*, 1982) can thus be reduced to the boson theory. In particular there is a fermionic Itô formula analogous to (2.3) in which the basic Itô multiplication table is identical to (2.5):

	$d\Lambda$	$dB^\dagger$	dB	ds
$d\Lambda$	$d\Lambda$	$dB^\dagger$	0	0
$dB^\dagger$	0	0	0	0
dB	dB	ds	0	0
ds	0	0	0	0

(2.6)

This is supplemented by the rules (2.4a) and (2.4c), but, instead of (2.4b) we find

$$E\,dB = dB\,E^\gamma, \qquad E\,dB^\dagger = dB^\dagger\,E^\gamma \tag{2.7}$$

where γ denotes conjugation by the self-adjoint unitary parity process:

$$E^\gamma = \Gamma E \Gamma = \Gamma E \Gamma^{-1}$$

3. FERMIONIC STOCHASTIC EVOLUTIONS

Let there be given a Hilbert space $\mathscr{H}_0$, called the *initial space*, which is the carrier Hilbert space for describing some physical system. The boson stochastic calculus is easily extended to integrands which are adapted operator valued processes on the domain $\mathscr{H}_0 \underline{\otimes} \mathscr{E}$ in $\mathscr{H}_0 \otimes \Gamma(h)$, by identifying the basic processes $A^\dagger$, Λ, and A with their ampliations $1_{\mathscr{H}_0} \underline{\otimes} A^\dagger$, $1_{\mathscr{H}_0} \underline{\otimes} \Lambda$, $1_{\mathscr{H}_0} \otimes A$. To make a similar extension in the fermion case, we assume $\mathscr{H}_0$ is equipped with a self-adjoint unitary *initial parity operator* Γ_0 and decompose $\mathscr{H}_0$ into its *even* and *odd* subspaces $\mathscr{H}_{0\pm}$, which are eigenspaces of Γ_0 corresponding to eigenvalues ± 1, respectively. We identify fermion creation and annihilation processes with their Z_2*-graded ampliations* to $\mathscr{H}_0 \underline{\otimes} \mathscr{E}$, defined by

$$1 \,\underline{\hat{\otimes}}\, B(t) u \otimes \phi = \mp u \otimes B(t)\phi \qquad (u \in \mathscr{H}_0,\ \phi \in \mathscr{E})$$

where the sign is $+$ or $-$ according as u (assumed of definite parity) belongs to $\mathscr{H}_{0+}$ or $\mathscr{H}_{0-}$.

We may consider as quantum stochastic generalizations of Schrödinger evolutions unitary *stochastic evolutions* driven by fermionic stochastic differential equations

$$dU = U(l_1\, d\Lambda + dB^\dagger\, l_2 + l_3\, dB + l_4\, ds), \qquad U_0 = 1 \tag{3.1}$$

Here the l_j are operators in $\mathscr{H}_0$ identified with their Z_2-graded ampliations to $\mathscr{H}_0 \otimes \Gamma(h)$. Necessary conditions for unitarity, assuming boundedness, are found, by equating to zero the stochastic differentials of $U^\dagger U$ and $UU^\dagger$, to be

$$l_1^* + l_1 + l_1^* l_1 = l_1 + l_1^* + l_1 l_1^* = 0$$
$$l_3^* + l_2 + l_1^{\gamma *} l_2 = l_2 + l_3^* + l_1^\gamma l_3^* = 0$$
$$l_2^* + l_3 + l_2^* l_1^\gamma = l_3 + l_2^* + l_3 l_1^{\gamma *} = 0$$
$$l_4^* + l_4 + l_2^* l_2 = l_4 + l_4^* + l_3 l_3^* = 0$$

where γ is the parity automorphism obtained by conjugating by Γ_0. Equivalently, $(l_1, l_2, l_3, l_4) = (w - 1, l, -l^* w^\gamma, ih - \frac{1}{2} l^* l)$, where w is unitary, h self-adjoint, and l arbitrary in $B(\mathscr{H}_0)$. This may be compared with the corresponding condition for the boson case (Hudson and Parthasarathy, 1986). If U is required to be even, then w and h must be even and l odd and the boson and fermion conditions for unitarity become formally identical.

It was proved in Applebaum and Hudson (1984) in the case when there is no preservation term, using a different formulation of fermionic stochastic calculus, in Hudson and Parthasarathy (1986) in the case when the initial grading is trivial, $\Gamma_0 = 1$, and in Applebaum (1987) in the general case, that these conditions are sufficient for the existence of a unique unitary process satisfying (3.1). Recently there has been considerable progress in the boson case in the theory of stochastic evolutions driven by unbounded driving coefficients (Applebaum, 1991; Mohari, 1991; Fagnola, 1990; Vincent-Smith, 1991).

4. FERMION FLOWS

By analogy with the boson theory (Hudson, 1990), a *fermion flow* is a generalization of a Heisenberg evolution incorporating fermion noise terms, described by a system of stochastic differential equations of the form

$$dj(x) = j(\lambda(x))\, d\Lambda + dB^\dagger\, j(\beta(x)) + j(\beta^\dagger(x))\, dB + j(\tau(x))\, ds, \quad j_0(x) = x \otimes 1 \tag{4.1}$$

Here each j_t is an injective homomorphism into $B(\mathscr{H} \otimes \Gamma(h))$ from a unital C*-subalgebra $\mathscr{A}$ of $B(\mathscr{H}_0)$, which we assume is invariant under the parity automorphism. Thus the family $j = (j_t : t \in \mathbb{R}_+)$ is a C*-quantum stochastic process in the sense of Accardi *et al.* (1982). Each $j(x)$, $x \in \mathscr{A}$, is an adapted process and λ, β, $\beta^\dagger$, and τ are maps from $\mathscr{A}$ into itself. In order to have such a system of stochastic differential equations, it is necessary to assume that j is even, in the sense that

$$j_t(x^\gamma) = j_t(x)^\gamma$$

[the consequences of abandoning this simplification are explored in Hudson and Shepperson (1992)].

As in the boson theory, unitality, linearity, and self-adjointness of the maps j_t require that the *structure maps* λ, β, $\beta^\dagger$, τ vanish on 1, are linear, and satisfy

$$\lambda(x^*) = \lambda(x)^*, \qquad \beta(x^*) = \beta^\dagger(x)^*, \qquad \tau(x^*) = \tau(x)^*, \qquad x \in \mathscr{A} \tag{4.2}$$

By differentiating the multiplicativity condition $j(xy) = j(x)j(y)$, we obtain the structure relations

$$\lambda(xy) = \lambda(x)y + x\lambda(y) + \lambda(x)\lambda(y) \tag{4.3a}$$

$$\beta(xy) = \beta(x)y + x^\gamma\beta(y) + \lambda(x)^\gamma\beta(y) \tag{4.3b}$$

$$\beta^\dagger(xy) = \beta^\dagger(x)y^\gamma + x\beta^\dagger(y) + \beta^\dagger(x)\lambda(y)^\gamma \tag{4.3c}$$

$$\tau(xy) = \tau(x)y + x\tau(y) + \beta^\dagger(x)\beta(y) \tag{4.3d}$$

A further set of structure relations follows from the evenness of j:

$$\lambda(x^\gamma) = \lambda(x)^\gamma, \quad \beta(x^\gamma) = -\beta(x)^\gamma, \quad \beta^\dagger(x^\gamma) = -\beta(x)^\gamma, \quad \tau(x^\gamma) = \Gamma(x)^\gamma \tag{4.4}$$

The basic existence theorem for fermion flows, parallel to that for the boson case (Evans, 1989), is as follows.

Theorem. Let λ, β, $\beta^\dagger$, τ be bounded linear maps vanishing on 1 and satisfying the structure relations (4.2)–(4.4). Then there exists a unique family of injective C*-algebra homomorphisms satisfying (4.1).

5. PERTURBATION THEORY

As in the boson case, we say j is *inner* if it is of form $j_t(x) = U_t x U_t^{-1}$ for some even unitary stochastic evolution U; equivalently, if its structure maps are of the form

$$\lambda(x) = wxw^{-1} - x$$

$$\beta(x) = lx - wx^\gamma w^* l$$

$$\beta^\dagger(x) = xl^* - l^* wx^\gamma w^*$$

$$\tau(x) = i[h, x] - \tfrac{1}{2}(l^* lx - 2l^* wx^\gamma w^* l + xl^* l)$$

where $w, l, h \in \mathscr{A}$, w is unitary and even, h is self-adjoint and even, and l is odd. More generally, given a unitary stochastic evolution driven by such w, l, and h, together with an arbitrary fermion flow j with structure maps λ, β, $\beta^\dagger$, and τ, we may construct a new flow $\tilde{j}$, called the perturbation of j by U, as follows. We first define a unitary process $u^{(j)}$ by

$$dU^{(j)} = U^{(j)}((j(w) - 1)\, d\Lambda + dB^\dagger j(l) - j(l^*w)\, dB + j(ih - \tfrac{1}{2}l^*l)\, ds),$$
$$U^{(j)}(0) = 1 \tag{5.1}$$

Then $\tilde{j}$ is given by

$$\tilde{j}_t(x) = U_t^{(j)} j(x) U_t^{(j)*} \tag{5.2}$$

Its structure maps are given by

$$\tilde{\lambda}(x) = w\lambda(x)w^{-1} + wxw^{-1} - x$$
$$\tilde{\beta}(x) = lx + w\beta(x) - w(x^\gamma + \lambda(x)^\gamma)w^*l$$
$$\tilde{\beta}^\dagger(x) = xl^* + \beta^+(x)w^* - l^*w(x^\gamma + \lambda(x)^\gamma)w^*$$
$$\tilde{\tau}(x) = \tau(x) + i[h, x] - \tfrac{1}{2}(l^*lx - 2l^*wx^\gamma w^*l + xl^*l)$$
$$- l^*w\beta(x) - \beta^\dagger(x)w^*l + l^*w\lambda(x)^\gamma w^*l$$

as may be verified by differentiating (5.2) using the quantum Itô formula. An existence, uniqueness, and unitarity theorem for solutions of (5.1) in the boson case is proved in Evans and Hudson (1990).

6. FERMION FLOWS AND SUPERSYMMETRY

In Jaffe *et al.* (1989) a *quantum algebra* is defined as a quadruple $(\mathscr{A}, \gamma, \beta, (\sigma_t : t \in \mathbb{R}))$, where $\mathscr{A}$ is a C*-algebra, γ is a $\mathbb{Z}_2$-grading involution of $\mathscr{A}$, and β is an odd γ-derivation of $\mathscr{A}$:

$$\beta(x^\gamma) = -\beta(x)^\gamma, \qquad \beta(xy) = \beta(x)y + x^\gamma\beta(y), \qquad x, y \in \mathscr{A}$$

which implies that β^2 is an even derivation:

$$\beta^2(xy) = \beta^2(x)y + x\beta^2(y)$$

and $(\sigma_t : t \in \mathbb{R})$ is a one-parameter group of automorphisms of the Banach algebra $\mathscr{A}$ (in general not *-automorphisms) of which β^2 is the infinitesimal generator. Assuming a 1–1 correspondence between such one-parameter groups and generators, which will hold, for example, if the latter are bounded, we may specify the quantum algebra by the triple $(\mathscr{A}, \gamma, \beta)$ alone.

In order to embed such a structure in a fermion flow (in which there will be no conservation term, hence only two independent structure maps β and τ), we must find an even linear map τ satisfying (4.3d). Equivalently, the map

$$\eta_\beta\colon \quad \mathscr{A} \times \mathscr{A} \to \mathscr{A}, \qquad \eta_\beta(x, y) = \beta^\dagger(x)\beta(y)$$

where $\beta^\dagger(x) = \beta(x^*)^*$, must be a 2-coboundary in the Hochschild cohomology of $\mathscr{A}$ with $\mathscr{A}$ itself as bimodule and left and right actions by multiplication. In fact η_β is automatically a 2-cocycle because of the twisted derivation property of β and it is a coboundary if β is inner (Hudson, 1987). Furthermore, every odd γ-derivation β is inner provided that the grading automorphism γ is also inner (Davies and Lindsay, n.d.).

Generalizing the notion of supertrace, Jaffe *et al.* (1989) define a *super KMS-functional* to be a continuous linear map $\omega\colon \mathscr{A} \to \mathbb{C}$ satisfying $\omega \circ \beta = 0$ and $\omega(xy) = \omega(y^\gamma \sigma_i(x))$, $\forall x, y \in \mathscr{A}$. Such a functional is even and satisfies

$$\omega(x\beta(y)) = -\omega(\beta(x)y^\gamma)$$

Thus it is of interest to know whether such a functional can be constructed for a fermion flow.

In Jaffe *et al.* (1989) a *Chern character* is defined for such a functional, and shown to be invariant under perturbation of β by inner γ-derivations:

$$\beta \mapsto \tilde{\beta} = \beta + ad_q^\gamma, \qquad ad_q^\gamma(x) = qx - x^\gamma q$$

It is natural to seek to define such a Chern character directly for fermion flows, and to prove its invariance under the perturbation theory of Section 5.

ACKNOWLEDGMENTS

Parts of this work were completed while the author visited MIEM, Moscow, the Mathematical Institutes of the Slovak Academy of Sciences, Bratislava, and the University of Łódź in autumn 1991. The hospitality of these institutions is gratefully acknowledged.

NOTE ADDED IN PROOF

Since this work was completed, the following related paper has appeared. Applebaum, D. B. (1993), Fermionic stochastic differential equations and the index of Fredholm operators, *Letters in Mathematical Physics*, **28**, 231–237.

REFERENCES

Accardi, L., Frigerio, A., and Lewis, J. T. (1982). Quantum stochastic processes, *PRIMS Kyoto*, **18**, 97–133.

Applebaum, D. B. (1987). Fermion Itô's formula II: The gauge process in fermion Fock space, *PRIMS Kyoto*, **23**, 17–56.

Applebaum, D. B. (1991). Unitary evolutions and horizontal lifts in quantum stochastic calculus, *Communications in Mathematical Physics*, **140**, 63–80.

Applebaum, D. B., and Hudson, R. L. (1984). Fermion Itô's formula and stochastic evolutions, *Communications in Mathematical Physics*, **96**, 473–496.

Barnett, C., Streater, R. F., and Wilde, I. F. (1982). The Itô–Clifford integral, *Journal of Functional Analysis*, **48**, 172–212.

Cockroft, A. M., and Hudson, R. L. (1977). Quantum mechanical Wiener processes, *Journal of Multivariate Analysis*, **7**, 107–124.

Davies, E. B., and Lindsay, J. M. (1993). Superderivation and symmetric Markov semigroups, *Communications in Mathematical Physics*, **157**, No. 2.

Evans, M. P. (1989). Existence of quantum diffusions, *Probability Theory and Related Fields*, **81**, 473–483.

Evans, M. P., and Hudson, R. L. (1990). Perturbations of quantum diffusions. *Journal of the London Mathematical Society (2)*, **41**, 373–384.

Fagnola, F. (1990). On quantum stochastic differential equations with unbounded coefficients, *Probability Theory and Related Fields*, **86**, 501–516.

Hudson, R. L. (1987). Quantum diffusions and cohomology of algebras, in *Proceedings of the 1st World Congress of the Bernoulli Society (Tashkent, 1986)*, Yu. Prohorov *et al.*, eds., VNU Press, Volume 1, pp. 479–485.

Hudson, R. L. (1990). Quantum stochastic flows, in *Probability Theory and Mathematical Statistics* (Proceedings, Vilnius 1989), B. Grigelionis *et al.*, eds., VSP, Volume 1, pp. 512–525.

Hudson, R. L., and Parthasarathy, K. R. (1984). Quantum Ito's formula and stochastic evolutions, *Communications in Mathematical Physics*, **93**, 301–323.

Hudson, R. L., and Parthasarathy, K. R. (1986). Unification of boson and fermion stochastic calculus, *Communications in Mathematical Physics*, **104**, 457–470.

Hudson, R. L., and Shepperson, P. (1992). Structure relations for fermion flows, in *Quantum Probability VII*, L. Accardi *et al.*, eds., World Scientific, Singapore.

Jaffe, A., Lesniewski, A., and Wisniowski, M. (1989). Deformations of super KMS functionals, *Communications in Mathematical Physics*, **121**, 527–540.

Mohari, A. (1991). Quantum stochastic differential equations with unbounded coefficients, ISI New Delhi preprint.

Parthasarathy, K. R. (1992). *An Introduction to Quantum Stochastic Calculus*, Birkhäuser, Basel.

Vincent-Smith, G. F. (1991). Unitary quantum stochastic evolutions, *Proceedings of the London Mathematical Society*, **63**, 1–25.